A Simplified Analytic CAD Model for Linearly Tapered Microstrip Lines

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Abstract - **Quasi-TEM propagation, appropriate for a linearly tapered microstrip line (LTML), is modeled and known microstrip impedance behavior is approximated directly. The telegrapher's equation and the resulting ABCD matrix is solved in terms of Airy functions, commonly available in scientific programming libraries. The theoretical model has been verified with measurements.**

I. Introduction

Linearly tapered microstrip lines (LTML) are important for matching networks, eliminating step discontinuities between transmission lines and lumped elements such as transistors, etc. They are also used in analog signal processing and pulse shaping and are a common component of VLSI design and allow for smoother connections between high-density integrated circuits.

Though much work has been done in the area of non-uniform transmission lines, a simplified analytical model for a linearly tapered microstrip transmission line (width varying linearly with longitudinal distance), employing appropriate quasi-static propagation assumptions has not been previously developed and supported by experimental results. This paper presents such a model. The early work as illustrated by [1] and [2], though mathematically elegant, made ideal transmission line assumptions such as TEM propagation and examined situations where the distributed impedance and admittance were contrived so that an exact analytic solution would be possible for the telegrapher's equations.

While still assuming TEM propagation [3] analyzed a non-uniform transmission line where the distributed impedance was linearly tapered as a function of longitudinal distance. The tapered line was viewed as a two port circuit and the ABCD matrix parameters were presented in terms of Bessel functions, albeit incorrectly. In [4] non-uniform transmission lines were analyzed in terms of cascaded linearly tapered lines, which included a correction to the previous Bessel function ABCD matrix model. In this case the TEM propagation in each cascaded section combines to approximate the quasi-TEM propagation in the complete non-uniform line. A linearly tapered microstrip line (linearly tapered width) was analyzed using ten sections and results were equivalent to those produced by a twenty section uniform line approximation. No models were compared against measured experimental results.

In this work, an approximate closed form for the solution of a LTML is found. The characteristic impedance and effective dielectric constant as a function of W/H ratio are examined for typical microstrip lines. It is noted that these properties can be closely approximated by exponential and linear expressions. When these expressions are used in the telegrapher's equation for a linear W/H variation, a solution is found in terms of Airy functions, which are commonly available in scientific programming libraries. Experimental results are shown to verify the resulting model.

II. Model Development

We begin by treating a linearly tapered microstrip line (LTML) of length ℓ as a loss-less transmission line whose distributed series impedance and distributed shunt admittance is represented by pure inductance and capacitance, respectively. Both the inductance and capacitance are functions of the distance along the microstrip line, x. A time harmonic dependence, i.e. $e^{j\omega t}$, is assumed and results in equations (1) and (2).

$$Z(x) = j\omega L(x) \tag{1}$$

$$Y(x) = j\omega C(x) \tag{2}$$

The telegrapher's equations, (3) and (4) express the relationship between voltage and current.

$$\frac{dI(x)}{dx} = -Y(x)V(x) \tag{3}$$

$$\frac{dV(x)}{dx} = -Z(x)I(x) \tag{4}$$

Differentiating (4) and substituting (3) yields

$$\frac{d^2V}{dx^2} - \left(\frac{1}{Z}\frac{dZ}{dx}\right)\frac{dV}{dx} - (YZ)V = 0 \tag{5}$$

The unit-less quantity YZ is recognized as the propagation constant squared and is related to the dielectric constant. For quasi-TEM propagation ε_r is replaced with the effective dielectric constant ε_{eff}.

$$YZ = -\omega^2 LC = -(\omega / v_p)^2 = -(\omega / c)^2 \varepsilon_{eff} \qquad (6)$$

Variation of microstrip effective dielectric constant and characteristic impedance has been modeled accurately using the Hammerstad and Jensen [5] model shown below in Fig. 1 for a range of dielectric constants representing commonly used substrates. One immediately notes that the characteristic impedance and effective relative dielectric constant for a LTML on a dielectric substrate, can be approximated with exponential and linear expressions as a function of distance. These expression in turn impose a functional form on the distributed series impedance and the distributed admittance of the line. Such plots motivate the assumptions made in the development of the LTML model that follows.

Motivated by the apparent linear nature of ε_{eff}, we let $\varepsilon_{eff} = d_1 + d_2 x$ for $0 \le x \le \ell$, where ℓ is the length of the tapered line. Therefore

$$YZ = -(\omega / c)^2 \varepsilon_{eff} = -(\omega / c)^2 (d_1 + d_2 x) \qquad (7)$$

Since the characteristic impedance for a LTML is observed to be exponential in nature we are motivated to

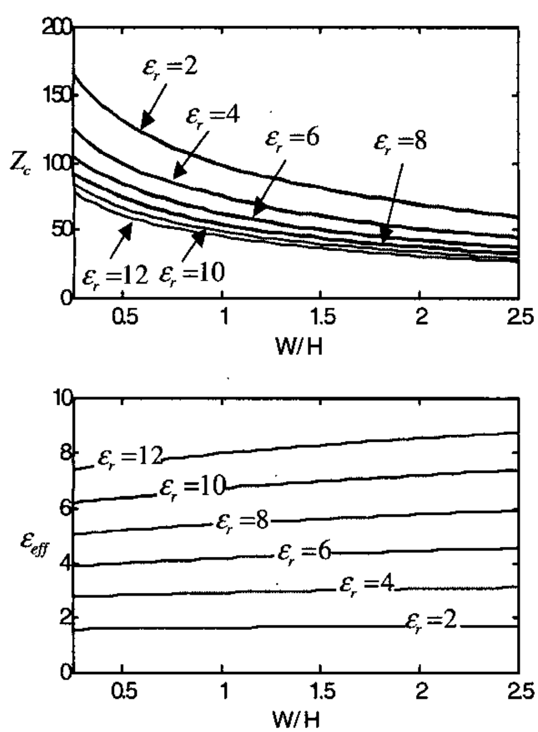

Fig. 1. Microstrip characteristic impedance and effective dielectric constant using Hammerstad and Jensen model.

consider letting $Z(x)$ be of the form

$$Z = j\omega d_3 e^{-d_4 x} \qquad (8)$$

Since the characteristic impedance equals,

$$Z_c = \sqrt{Z/Y} = \sqrt{Z^2/YZ}\,,$$

then

$$Z_c = c d_3 \frac{e^{-d_4 x}}{\sqrt{d_1 + d_2 x}} \qquad (9)$$

We note that the characteristic impedance is dominated by an exponential behavior as desired. Substituting (7) and (8) in the second order differential equation (5) yields (10).

$$\frac{d^2 V}{dx^2} + A\frac{dV}{dx} + (B + Cx)V = 0 \qquad (10)$$

where $0 \le x \le \ell$, and

$$A = d_4,\ B = (\omega / c)^2 d_1, \text{ and } C = (\omega / c)^2 d_2$$

The solution to this differential equation is expressed in terms of the Airy Functions of the first and second kind (see IV. APPENDIX) as shown below.

$$V(x) = C_1 F(x) + C_2 G(x) \qquad (11)$$

where

$$F(x) = e^{-\frac{Ax}{2}} Ai(\zeta) \qquad (12)$$

$$G(x) = e^{-\frac{Ax}{2}} Bi(\zeta) \qquad (13)$$

$$\zeta = \frac{1}{4}\frac{A^2 - 4B - 4Cx}{(-C)^{2/3}} \qquad (14)$$

The terms C_1 and C_2 are the arbitrary constants that accompany a general solution of a second order differential equation. Ai and Bi are Airy functions of the first and second kind, respectively. The current is found using the telegrapher equation (4), which results in

$$Z(x)I(x) = -C_1 F'(x) - C_2 G'(x) \qquad (15)$$

Direct substitution of (11) and (15) yields

$$\begin{pmatrix} V(x) \\ Z(x)I(x) \end{pmatrix} = \begin{pmatrix} F(x) & G(x) \\ -F'(x) & -G'(x) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad (16)$$

Letting M and U be defined as

$$M(x) = \begin{pmatrix} F(x) & G(x) \\ -F'(x) & -G'(x) \end{pmatrix} \qquad (17)$$

and

$$U(x) = \begin{pmatrix} 1 & 0 \\ 0 & Z(x) \end{pmatrix} \qquad (18)$$

and letting

$$E(x) = M(x)^{-1} U(x) \tag{19}$$

then (16) implies that the following matrix product is invariant, i.e., a constant, or independent of the distance x.

$$E(x)\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} \tag{20}$$

If $E_1 = E(x=0)$, and $E_2 = E(x=\ell)$ then it follows that

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = E_1^{-1} E_2 \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}, \tag{21}$$

which we recognize as the ABCD matrix describing a two port circuit. Therefore,

$$\mathrm{ABCD} = E_1^{-1} E_2 \tag{22}$$

III. Model Verification

Four circuits, designated Circuit A through D, were constructed to verify the Airy function models. Fig. 2 illustrates two types of circuits used for model verfication (see Table I). TRL calibration and de-embeding was employed with an 8510 to obtain the results shown in Fig. 3. For the circuits each LTML was modeled as two cascaded tapered sections. ABCD matrices for each section were then modeled using the new Airy function formalism.

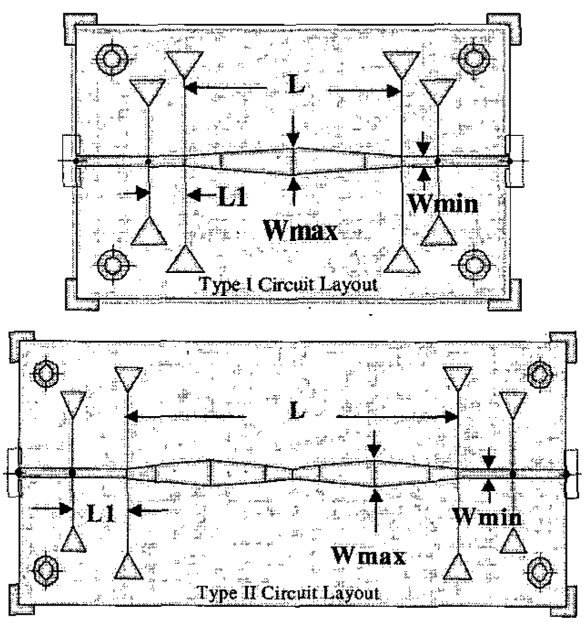

Fig. 2. Microstrip circuits used for experimental verification of the Airy function modeling

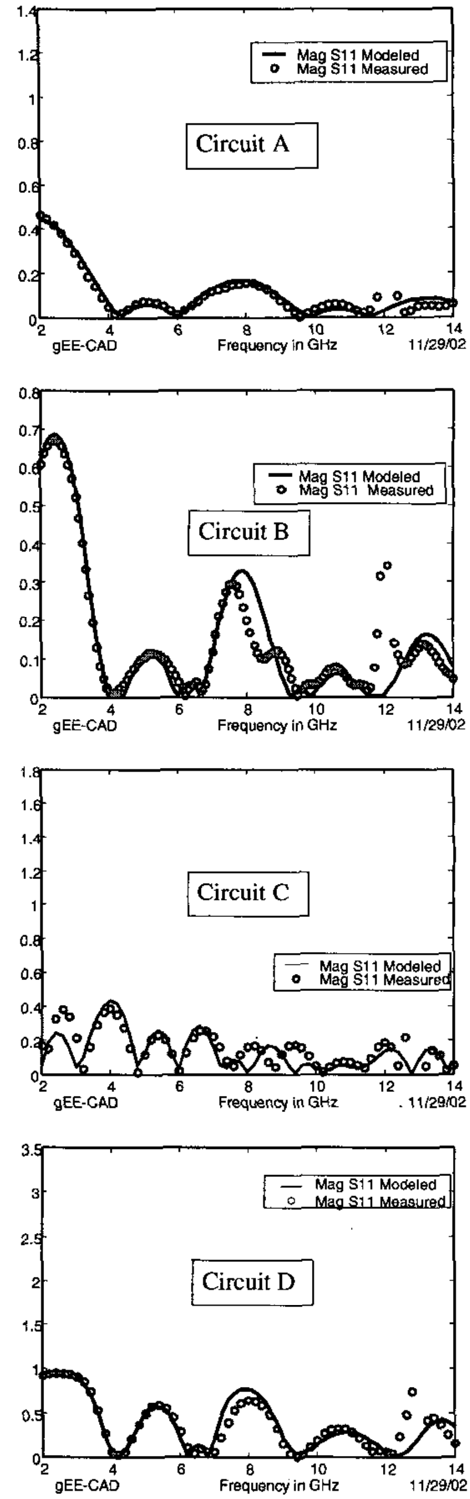

Fig. 3. Comparison of measured versus modeled data for test circuits A, B, C, and D (See Fig. 2 and Table I).

TABLE I*

Circuit ==>	A	B	C	D
ε_r	2.6	2.6	9.8	9.8
H	30	30	25	25
W max	200	200	200	200
W min	81.7	81.7	23.6	23.6
L	1500	3000	1500	1500
L1	250	500	250	250
CKT Type	I	II	I	II

*All lengths in mils.

As further verification, results from our approach using the Airy function model have been compared to those obtained using an iterative impedance transformer method. In this method a tapered line is divided into multiple equal-length short segments with impedances that vary stepwise from one segment to another. In this case the input impedance is determined for the combined set of lines and then used to calculate the reflection coefficient (S_{11}) or other S parameters. Since one expects a larger number of segments to produce a more accurate approximation the tapered line was divided into 1000 segments. This is computationally slow, but a comparison showed that results from this model were identical with those obtained using the Airy function method.

IV. Appendix

The voltage differential equation (10) is of the following form where A is a constant and $f(x)$ is a linear function.

$$\frac{d^2V}{dx^2}+A\frac{dV}{dx}+f(x)V=0 \qquad (23)$$

The first derivative term can be eliminated introducing a function $W(x)$ defined by the transformation

$$V(x)=e^{-\frac{Ax}{2}}W(x) \qquad (24)$$

The differential equation, where $f(x)=B+Cx$, then becomes

$$\frac{d^2W}{dx^2}-\left(\frac{A^2}{4}-B-Cx\right)W=0 \qquad (25)$$

Motivated by the fact that this resembles the Airy differential equation,

$$\frac{d^2u(\zeta)}{d\zeta^2}-\zeta u(\zeta)=0 \qquad (26)$$

we transform the independent variable using the relationship $\zeta=\beta+\gamma x$ expecting that the constants β and γ can eventually be chosen so that (25) turns into (26). Solving for x and substiuting the results into (25) produces

$$\gamma^2\frac{d^2W}{d\zeta^2}-\left(\frac{A^2}{4}-B+\frac{C\beta}{\gamma}-\frac{C\zeta}{\gamma}\right)W=0$$

The constant part of the expression in the parentheses vanishes if

$$\beta=\frac{\gamma}{C}\left(B-\frac{A^2}{4}\right) \qquad (27)$$

and the differential equation simplifies to

$$\gamma^2\frac{d^2W}{d\zeta^2}+\frac{C\zeta}{\gamma}W=0 \qquad (28)$$

Letting $\gamma=(-C)^{1/3}$, this becomes the Airy differential equation where

$$\beta=\frac{1}{(-C)^{2/3}}\left(\frac{A^2}{4}-B\right) \qquad (29)$$

The solution to the original differential equation (10) is therefore given by

$$V=e^{-\frac{Ax}{2}}W(\zeta) \qquad (30)$$

where

$$\zeta=\frac{\left(\frac{A^2}{4}-B\right)-Cx}{(-C)^{2/3}} \qquad (31)$$

and $W(\zeta)$ is an Airy function of either the first or second kind, usually denoted by Ai and Bi.

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